COSMOS Cluster 5

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Contents

1 Reflection and Refraction

Objective: Understand the behavior of light as it pass through different mediums.

1.1 Ray Approximation

Learn about the mathematical equations that describe reflection and refraction.

Light travels in a straight line in a homogeneous medium until it encounters a boundary between two different media. A **ray** of light is an imaginary line drawn along the direction of propagation of the light. A **wavefront** is a surface passing through points of a wave that have the same phase. The rays, corresponding to the direction of the wave motion, are perpendicular to the wavefronts

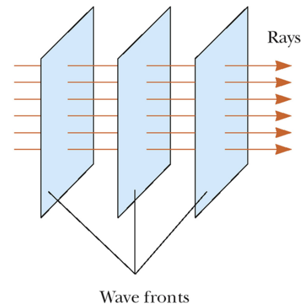
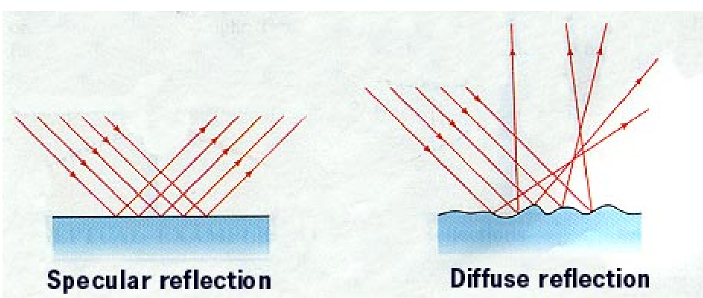


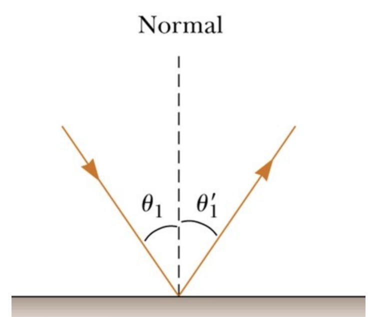
Figure 1: Ray approximations and wavefronts.

1.2 Reflection

When light encounters a boundary with a second medium, part of this incident light is directed back into the first medium, and this is known as **reflection**. If the boundary is a smooth surface, the reflection is known as specular reflection. This means all the reflected rays will be parallel to one another. Otherwise if the boundary is not smooth, then the reflection is called defuse reflection (See Figure 2a). The **normal** to a surface is a line which is perpendicular to the surface at given point. The **incident ray** makes an angle θ_1 with the normal. The **reflected ray** makes an angle θ_1' with the normal. The angle of reflection is equal to the angle of incidence: $\theta_1 = \theta_1'$ (See Figure 2b).



(a) Specular/Diffuse reflection



(b) Reflection

Figure 2: Depictions of reflection.

It is important to note that reflection always occurs when light interacts with different mediums.

1.3 Refraction

When light, propagating through a transparent medium, encounters a boundary leading into another transparent medium, part of the light is reflected, and part of the light passes into the second medium. The ray that enters the second medium is bent at the boundary. This bending of the ray is called **refraction**. The incident ray, the reflected ray, and the refracted ray (and the normal too) all lie in the same plane. The angle of refraction θ_2 depends on the refractive indices of the both media.

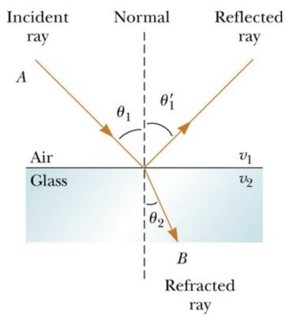


Figure 3: Refraction

The **angle of refraction** (θ_2 in Figure 3) depends on the material and the angle of incidence

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{v_1}{v_2}$$

The index of refraction, or **refractive index**, n, of a medium is a unitless value defined as

$$n = \frac{\text{speed of light in a vacuum}}{\text{speed of light in the medium}} = \frac{c}{v}$$

Note that for a vacuum, $v = c$, so $n = 1$. Otherwise, $n > 1$. We can derive **Snell's law** at this point.

Let n_1 and n_2 be the refractive index of the first and second medium, respectively.

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{v_1}{v_2} = \frac{c/n_1}{c/n_2} = \frac{n_2}{n_1} \implies n_1 \sin\theta_1 = n_2 \sin\theta_2$$

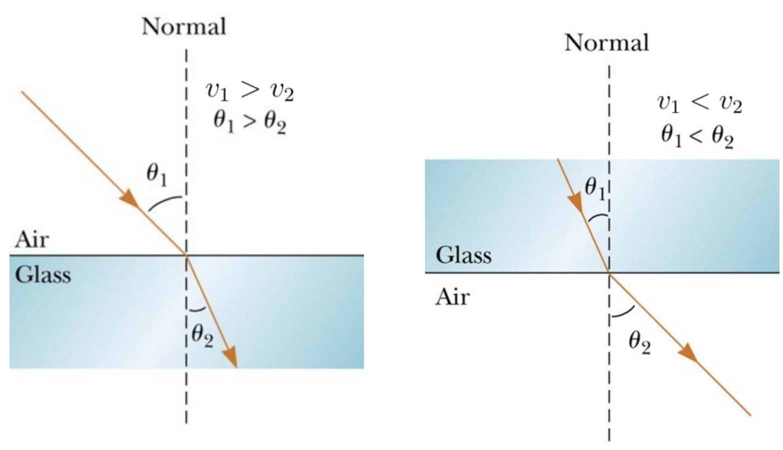
Consider the two cases in Figure 5. In Figure 5a, observe that the angle of refraction, θ_2, is less than the angle of incidence, θ_1. On the other hand, in Figure 5b, has the inequality relation to be flipped.

Simply put, in terms of index of refraction, we have that the medium with the greater index of refraction has a lower angle of refraction. It is important to understand the relationships between all of these values. Consider the following example. Let θ_1 and n_1 be constant. Then we have

$$\underbrace{n_1 \sin\theta_1}_{\text{constant}} = n_2 \sin\theta_2$$

Furthermore $\sin\theta \approx \theta$ for $0 \leq \theta \leq \frac{\pi}{2}$. Then we get that approximately $n_2\theta_2 = \eta$, for some constant η determined by n_1 and θ_1. What does this mean mathematically though? We can that given a fixed first medium and angle of incidence, the angle of refraction is inversely proportion to the refractive index.

With a similar strategy, we can show that θ_1 is directly proportional to θ_2 if n_1, n_2 are kept constant i.e. increasing θ_1 will increase θ_2 and decreasing θ_1 will decrease θ_2.



(a) $v_1 > v_2$ or $n_1 < n_2$

(b) $v_1 < v_2$ or $n_1 > n_2$

Figure 4: Two cases of refraction.

Figure 4 shows how the angle made with the normal is smaller in the medium with a higher refractive index. We will prove the result for Figure 4a. The proof for 4b is similar.

Proposition: If $n_1 < n_2$, then $\theta_1 > \theta_2$.

Proof: Recall the following proportions:

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{n_2}{n_1}$$

Since $n_2 > n_1 > 0$, so $\frac{n_2}{n_1} > 1$. Then $\frac{\sin\theta_1}{\sin\theta_2} > 1 \implies \sin\theta_1 > \sin\theta_2$. sin is a monotone increasing function in the domain $(0, \frac{\pi}{2})$. This means that for any x, y between 0 and $\frac{\pi}{2}$, if $\sin x > \sin y$ then $x > y$. We can now conclude that $\theta_1 > \theta_2$. ■

Consider the physical example in Figure 5, where we shine a laser through a prism. Ray 1 is the incident ray. Ray 2 is the reflected ray. Ray 3 is refracted into the crystal. Ray 4 is internally reflected in the crystal. Ray 5 is refracted as it enters the air from the crystal.

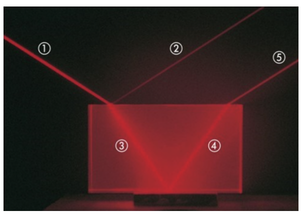


Figure 5: Light path of laser through a prism.

Another interesting example is to observe that Ray 2 and Ray 5 look parallel. It is left as an exercise to the reader to prove that these two rays are indeed parallel given that the prism is a perfect rectangular prism.

Just kidding, I will prove it here.

Proposition: Ray 2 and Ray 5 are parallel.

Proof: Let θ_1 denote the angle of incidence of the laser (Ray 1) and θ_2 denote the angle of the first reflection, Ray 2.

Let θ_3 be the refracted angle of Ray 3. Given that the top and bottom surfaces are parallel, this must mean that all lines perpendicular to these parallel lines are parallel to each other. We can draw all of the normals between Rays (1,2), (3,4), and (4,5), which is depicted with the magenta arrows.

Ray 3 is a transversal of the left and middle vertical lines. By the alternate interior angles theorem, $\theta_3 = \theta_r$ and $\theta_r' = \theta_4$. Furthermore, $\theta_r = \theta_r'$ by the properties of reflection of a smooth surface. By the transitive property and the congruent set of angles, we have $\theta_3 = \theta_4$.

We know that (θ_1, θ_3) and (θ_4, θ_5) satisfy Snell's law. But with what we had earlier, $n_2 \sin\theta_3 = n_2 \sin\theta_4$. So

$$n_1 \sin\theta_1 = n_2 \sin\theta_3 = n_2 \sin\theta_4 = n_1 \sin\theta_5 \implies \sin\theta_1 = \sin\theta_5$$

Since θ_1 and θ_5 can only be values between 0 and $\frac{\pi}{2}$, there must be unique values that satisfy this equality (in other words, we can say that the sin function is **bijective** in $(0, \frac{\pi}{2})$). Necessarily, we must have $\theta_1 = \theta_5$. With reflection, $\theta_1 = \theta_2 = \theta_5$.

Finally, let the top surface be a transversal. The angles of θ_2 and θ_5 are congruent, so Ray 2 and Ray 5 are parallel by the converse of the corresponding angles theorem. This concludes the proof. ■

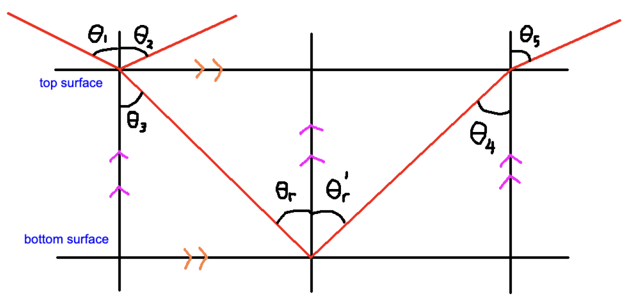


Figure 6: Depiction of laser's path through prism with labeled angles.

1.4 Wavelength and Frequency

Recall that the velocity of a wave is the product of the wavelength λ and frequency f i.e. $v = \lambda f$. When light travels through different mediums, what happens to these values?

The equation for the energy of a **photon**, a light particle, is $E = hf$, where h is Planck's constant. From this, we can deduce that the frequency of a wave remains constant as it travels through different mediums since energy must be conserved. With velocity changing and frequency remaining constant, we must have that the wavelength changes!

With this new idea in mind, we can derive another proportional relationship related to Snell's law. λ_1 and λ_2 are the wavelength of the light in the first and second medium.

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{v_1}{v_2} = \frac{\lambda_1 f}{\lambda_2 f} = \frac{\lambda_1}{\lambda_2}$$

Another point to note is that the velocity of light through a medium changes as wavelength changes. Essentially, velocity has a dependency on wavelength. The intuition regarding this relationship is from the fact that different wavelengths interact with the atoms in a different manner.

1.5 Total Internal Reflection

Total internal reflection can occur when light attempts to move from a medium with a high index of refraction to a medium with a lower index of refraction. A particular angle of incidence will result in angle of refraction of $90°$. This angle of incidence is called the critical angle. For angles of incidence greater than the critical angle, the beam is entirely reflected at the boundary. This ray obeys the Law of Reflection. Total internal reflection occurs only when light attempts to move from a medium of higher index of refraction to a medium of lower index of refraction.

1.6 Supplementary Materials

- 3blue1brown's video: '4 questions about the refractive index | Optics puzzles 4'

2 Diffraction

3 Spectrometer

4 Polarization

5 Solar Cell

6 Quantum Cryptography

6.1 Introduction to Cyptography

Cryptography is the technique for converting information to an unintelligible form that can only be deciphered by authorized parties. **Encrypting** is altering the data in a way to hide the true message. **Decrypting** is basically undoing the encryption to reveal the true message. A **key** is a piece of information such that it helps encrypt or decrypt the message. The goal is to only let authorized parties have access to the key. Thus, the encrypted message will make no sense to those without the key.

Especially in the digital world, we encode each letter to a specific binary string i.e. a sequence of eight 1's and 0's. Let's take a look at a simple example where we our encoding algorithm is simply adding a binary string to each character's respective binary string.

Before we proceed, let's do an example of adding binary numbers.

Example: Let the key be 11010100.

True message	N	O	Encrypted message	N	O

ASCII Alphabet			
A	1000001	N	1001110
B	1000010	O	1001111
C	1000011	P	1010000
D	1000100	Q	1010001
E	1000101	R	1010010
F	1000110	S	1010011
G	1000111	T	1010100
H	1001000	U	1010101
I	1001001	V	1010110
J	1001010	W	1010111
K	1001011	X	1011000
L	1001100	Y	1011001
M	1001101	Z	1011010

Part of the ASCII alphabet

Figure 7: Light path of laser through a prism.

6.2 Quantum Key Distribution

This time, we will make our key by using photons. According to quantum physics, one cannot create a copy of a single photon with the same state (polarization state in this case). This is based on the no-cloning theorem.

6.3 Linear Algebra Background

TO-DO

6.4 BB84 Protocol

The idea is to generate a secure key that only the sender and receiver have acces to. Single photons transmitted and measured at randomly selected polarization states are used to create a secure key.

Hotizontal and vertical polarizations make an orthogonal basis. -45 and $+45$ polarizations make an orthogonal basis.

In our experiment, a short pulse of laser light is used instead of a single photon. When a $45°$ light is sent through a polarizing beam splitter, both detectors will record light, In this case, one of the polarizations states will be randomly selectly electronically.

Alice will send a message in either $-45, 0, 45, 90$ using the waveplate. Bob will receive the message with a waveplate that can be configured to either $0, 45$. Bob will use the polarized beam splitter to project the lasers onto two different panels.

6.5 Additional Resources

- More cool cryptography videos
  - 'The Unbreakable Krytos Code' by LEMMiNO

– 'Cicada 3301:An Internet Mystery' by LEMMiNO

7 Waves

7.1 Wave Equation

A **wave** is a function of both **space** x and **time** t.

$$E = A\sin(\omega t - kx)$$

A is called the **amplitude** of the wave and has the same units as E. ω is the **angular-frequency** (rads/sec). If the frequency (in Hz) is f, then the angular frequency is $\omega = 2\pi f$. k is the **phase constant** or **wave number** (radians/meter). It is defined to be $k = \frac{2\pi}{\lambda}$, where λ is the wavelength.

Why is this equation important. Well, remember that the displacement from the equilibrium of a wave differs at every point in time and space. This equation captures all of that.

It may be challenging to think about this mutlivariable function. We can do the same thing as we did in Section 1, where we fix a specific variable. Note that we are work in a "3D" environment where the x-axis is space, y-axis is time, and z-axis is the distance from the equilibrium. Let's take an example wave $f(x, y) = \sin(x + \pi y)$.

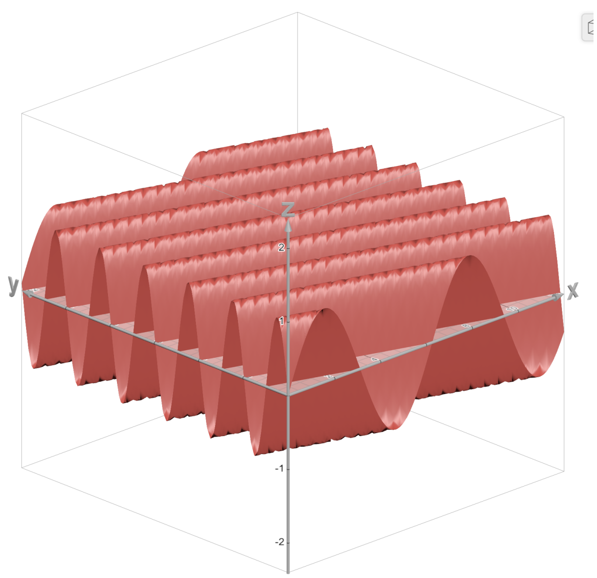


Figure 8: $f(x, y) = \sin(x + \pi y)$

Fix $x = 0$. Then we are taking a slice of the cubic cake that is perpendicular to the x-axis. This is the action of fixing an x. Mathematically, we have

$$f(0, y) = \sin(\pi y)$$

It is clear that the function is only a function of time. The function that we get will be the outlined curve on the left-hand-side (the thin sine wave). Convince yourself that the equation of that curve is $f(0, y) = \sin(\pi y)$. With this approach, we can see that fixing x, a point in space, we get a snapshot of how one point in space fluctuates.

We can do a similar approach by fixing time $t = 0$. Graphically, we are making a slice perpendicular to the y-axis and is essentially the outline of the curve on the right-hand-side surface.

Here is a physical example. I'm sure everyone has seen the rope being shaken up and down (in a sinusoidal manner) so that it looks like a wave. Imagine taking a photo of the shaking rope. That is fixing a point in time y. Now imagine looking at a specific section of the rope and just watching how that specific point of the rope changes in location. That is fixing a point in space x. In both scenarios.

Now imagine you change x, but still consider as a constant.

$$f(y) = \sin(\underbrace{x}_{\text{constant}} + \pi y)$$

x just behaves as a phase shift here. One way to think about this is that as we wake snapshots of the wave at

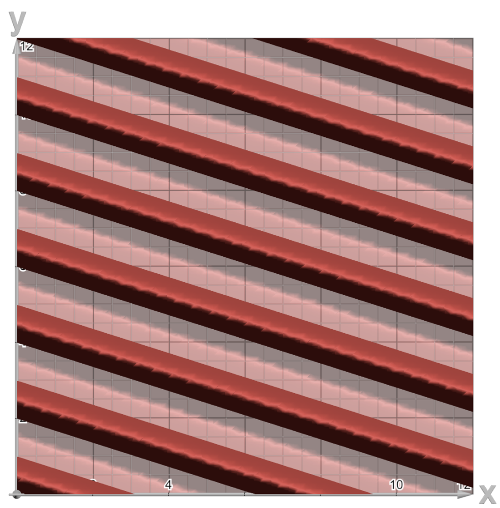


Figure 9: $f(x, y) = \sin(x + \pi y)$

The darker parts of the graph are the **crests**, or the highest point, of the wave, while the lighter parts are the **troughs**, or the lowest point.

Take a cut horizontally at $y = 0$, and consider the cross section. Then take another cut at $x = 0.01$. Then do it again for $x = 0.02$. This is like taking a 100 fps video of the shaking rope example. Observe that the crests of the wave are move towards the left. This is the **direction of propagation**. You can do a similar approach by fixing the cuts vertically to different points in time.

I just mostly dumped as many examples as possible to mathematically interpret waves. Hopefully this helped.

7.2 Polarization of Light

An EM wave is a vector equantity. That means that is has a magnitude and direction. Let's take a look at the general form of a wave fluctuations in a field.

$$\vec{E}(\vec{r}, t) = \vec{P} \cos(\omega t - \vec{k} \cdot \vec{r})$$

where $\cdot$ operation is the **dot product**. This the same exact deal with the previous section, but the position, amplitude, and wave numbers are vectorized. P is the polarization vector. k is the wave numbers in each coordinate direction, and $\vec{r}$ is the position vector.

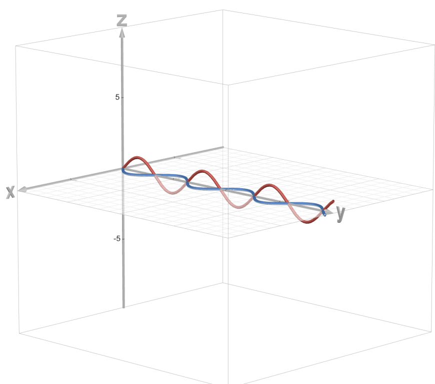
Let's take a simple example. Let let $\vec{P}$ be a 2-dimensional vector and $\vec{k}$, and $\vec{r}$ be scalars. This means that the wave propagates in 2-dimension.

$$\vec{E}(r,t) = \vec{P}\cos(\omega t - kr)$$

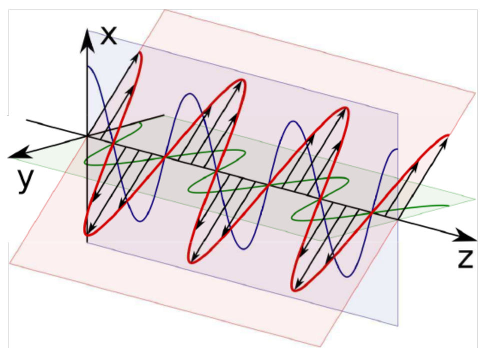
This looks pretty familiar. Let me break it down a little more. If $P = \begin{bmatrix} P_x \\ P_y \end{bmatrix}$, then

$$\begin{bmatrix} E_x \\ E_y \end{bmatrix} = \begin{bmatrix} P_x \cos(\omega t - kr) \\ P_y \cos(\omega t - kr) \end{bmatrix}$$

Observe that each component itself is just a normal wave, just as in the previous section. This equation characterizes the polarization of a wave with respect to time and multi-dimensional space.



(a) $\begin{bmatrix} x(y) \\ z(y) \end{bmatrix} = \begin{bmatrix} \sin y \\ \sin y \end{bmatrix}$



(b) Polarized light visualization.

In the figure above, we can see that there are fluctuations in the x and z-direction with a dependency on y. The second subfigure shows what the wave will look like when you add the red and blue components together. The slant of this curve is the **polarization**.

An electric field that has uniform properties at all points across an infinite plane (does not vary on that plane at any time point), is called a plane wave.

8 Stuff

Euler's Formula $e^{i\theta}$.

Linear polarized light can be represented as

$$\vec{E} = E_0 \cos(\omega t - kx) = \Re\{E_0 a_x e^{i\omega t - kx}\}$$

where $\Re$ denotes the real part of the complex number.

$$\vec{E} = E_0 a x_s e^{i(\omega t - kx)} = E_0 a x_s e^{i(\omega t - k_0 n x)}$$

And $\begin{bmatrix} e^{ik_0 n_f L} \\ e^{ik_0 n_s L} \end{bmatrix}$.