Chapter 1 - Curves

Jay Paek

June 27, 2024

Notation

- $\alpha : I \to \mathbb{R}^n$ is a parameterized curve from an interval to $\mathbb{R}^n$. The domain need not be bounded.
- If $\alpha : \mathbb{R} \to \mathbb{R}^2$, then $\alpha(t) = (x(t), y(t))$. If $\alpha : \mathbb{R} \to \mathbb{R}^3$, then $\alpha(t) = (x(t), y(t), z(t))$. Of course, $x, y, z : \mathbb{R} \to \mathbb{R}$.
- If $\alpha : \mathbb{R} \to \mathbb{R}^n$ and $n > 3$, then $\alpha_i : I \to \mathbb{R}$ is the ith component of the parametrized curve.
- $|.| : \mathbb{R}^n \to \mathbb{R}$ is the Euclidean norm, unless specified otherwise.

1-2 Exercises

Question 1

Find a parametrized curve $\alpha(t)$ whose trace is the circle $x^2 + y^2 = 1$ such that $\alpha(t)$ runs clockwise around the circle with $\alpha(0) = (0, 1)$.

Solution: $\alpha(t) = (\sin(t), \cos(t))$

Question 2

Let $\alpha(t)$ be a parametrized curve which does not pass through the origin. If $\alpha(t_0)$ is the point of the trace of α closest to the origin and $\alpha'(t_0) \neq 0$, show that the position vector $\alpha(t_0)$ is orthogonal to $\alpha'(t_0)$.

Solution:

$$\begin{aligned} \frac{d}{dt}\left(|\alpha(t)|^2\right) &= \frac{d}{dt}\Big(\sum_{i=1}^{n} \alpha_i(t)^2\Big) \\ &= \sum_{i=1}^{n} 2\alpha_i(t)\alpha_i'(t) \\ &= 2\langle \alpha(t), \alpha'(t) \rangle \end{aligned}$$

$|\alpha(t)|$ is a minimum at $t = t_0$, so $\dfrac{d}{dt}\left(|\alpha(t)|^2\right) = 0$.

Question 3

A parametrized curve $\alpha(t)$ has the property that its second derivative $\alpha''(t)$ is identically zero. What can be said about α?

Solution: A simple calculation yields that $\frac{d}{dt}\left(|\alpha'(t)|^2\right) = 2\langle \alpha'(t), \alpha''(t)\rangle = 0$. So the speed of the curve is constant.

Question 4

Let $\alpha : I \to \mathbb{R}^3$ be a parametrized curve and let $v \in \mathbb{R}^3$ be a fixed vector. Assume that $\alpha'(t)$ is orthogonal to v for all $t \in I$ and that $\alpha(0)$ is also orthogonal to v. Prove that $\alpha(t)$ is orthogonal to v for all $t \in I$.

Solution: Without loss of generality, pick $i = 1, 2, 3$.

$$\frac{d}{dt}\langle \alpha_i(t), v\rangle = \langle \alpha_i'(t), v\rangle = 0$$

Let $t \in I$. By the Fundamental Theorem of Calculus:

$$\langle \alpha_i(t), v\rangle = \int_0^t \langle \alpha_i'(\tau), v\rangle d\tau + \langle \alpha(0), v\rangle = 0$$

Question 5

Let $\alpha : I \to \mathbb{R}^3$ be a parametrized curve, with $\alpha'(t) \neq 0$ for all $t \in I$. Show that $|\alpha(t)|$ is a nonzero constant if and only if $\alpha(t)$ is orthogonal to $\alpha'(t)$ for all $t \in I$.

Solution: ($\Longrightarrow$): Let $t \in I$. $|\alpha(t)| \neq 0$ is a constant function that is differentiable if and only if

$$\frac{d}{dt}\left(|\alpha(t)|^2\right) = 2\langle \alpha(t), \alpha'(t)\rangle = 0 \iff \langle \alpha(t), \alpha'(t)\rangle = 0$$

With the chain of necessary and sufficient conditions, we are done.

1-3 Exercises

Question 1

Show that the tangent lines to the regular parametrized curve $\alpha(t) = (3t, 3t^2, 2t^3)$ make a constant angle with the line $y = 0$, $z = x$.

Solution: The tangent line of the parametrized curve is $\alpha'(t) = (3, 6t, 6t^2)$. Pick a point on the given line: $\mathbf{v} = (1, 0, 1)$.

$$\cos\theta = \frac{\alpha'(t) \cdot \mathbf{v}}{|\alpha'(t)||\mathbf{v}|} = \frac{3 + 6t^2}{\sqrt{2}\sqrt{9 + 36t^2 + 36t^4}} = \frac{1}{\sqrt{2}}$$

And thus θ must be constant w.r.t. $t = \dfrac{\pi}{2}$.

Note: we use the regularity of the curve when we divide by $|\alpha'(t)|$.

Question 2

A circular disk of radius 1 in the plane xy rolls without slipping along the x axis. The figure described by a point of the circumference of the disk is called a cycloid.

Part (a)

Obtain a parametrized curve $\alpha : \mathbb{R} \to \mathbb{R}^2$ the trace of which is the cycloid, and determine its singular points.

Solution: $\alpha(t) = (t - \sin t, 1 - \cos t)$. $\alpha'(t) = (1 - \cos t, \sin t)$. The singular points are when $|\alpha'(t)| = 0$.

$$|\alpha'(t)| = 1 - 2\cos t + \cos^2 t + \sin^2 t = 2 - 2\cos t = 0 \implies t = 2\pi n, n \in \mathbb{Z}$$

Part (b)

Compute the arc length of the cycloid corresponding to a complete rotation of the disk.

Solution: $\displaystyle\int_0^{2\pi} |\alpha'(t)| dt = \int_0^{2\pi} (2 - 2\cos t) dt = (2t - 2\sin t)_0^{2\pi} = 4\pi$

Question 3

Let $0A = 2a$ be the diameter of a circle S^1 and $0y$ and AV be the tangents to S^1 at 0 and A, respectively. A half-line r is drawn from 0 which meets the circle S^1 at C and the line AV at B. On $0B$ mark off the segment $0p = CB$. If we rotate r about 0, the point p will describe a curve called the *cissoid of Diocles*. By taking $0A$ as the x axis and $0Y$ as the y axis, prove that

Part (a)

The trace of

$$\alpha(t) = \left(\frac{2at^2}{1 + t^2}, \frac{2at^3}{1 + t^2} \right), \quad t \in \mathbb{R},$$

is the cissoid of Diocles ($t = \tan\theta$).

Solution: Let the slope of the half-line r be $t = \tan\theta$. Compute when r intersects S^1. The equation of the upper semicircle is $y = \sqrt{a^2 - (x-a)^2} = \sqrt{2ax - x^2}$

$$\begin{aligned} tx &= \sqrt{2ax - x^2} \\ t^2x^2 &= 2ax - x^2 \\ (1+t^2)x^2 &= 2ax \\ (1+t^2)x &= 2a \\ x &= \frac{2a}{1+t^2} \end{aligned}$$

Let $\alpha(t) = (x(t), y(t))$. We can obtain $x(t)$, and then $y(t)$:

$$\begin{aligned} x(t) &= 2a - x = \frac{2at^2}{1+t^2} \\ y(t) &= tx(t) = \frac{2at^3}{1+t^2} \end{aligned}$$

Part (b)

The origin $(0,0)$ is a singular point of the cissoid.

Solution:

$$\begin{aligned} x'(t) &= \frac{4at(1+t^2) - 2at^2(2t)}{(1+t^2)^2} = \frac{4at}{(1+t^2)^2} \\ y'(t) &= \frac{6at^2(1+t^2) - 2at^3(2t)}{(1+t^2)^2} = \frac{2at^4 + 6at^2}{(1+t^2)^2} \\ x'(0) &= 0, y'(0) = 0 \implies |\alpha'(t)| = 0 \end{aligned}$$

Part (c)

As $t \to \infty$, $\alpha(t)$ approaches the line $x = 2a$, and $\alpha'(t) \to (0, 2a)$. Thus, as $t \to \infty$, the curve and its tangent approach the line $x = 2a$; we say that $x = 2a$ is an asymptote to the cissoid.

Solution:

$$\lim_{t\to\infty} x(t) = 2a, \lim_{t\to\infty} y(t) = \infty, \lim_{t\to\infty} x'(t) = 0, \lim_{t\to\infty} y'(t) = 2a$$

Question 4

Let $\alpha : (0, \pi) \to \mathbb{R}^2$ be given by

$$\alpha(t) = \left(\sin t, \cos t + \log\tan\frac{t}{2}\right),$$

where t is the angle that the y axis makes with the vector $\alpha'(t)$. The trace of α is called the *tractrix* (Fig. 1-9). Show that

Part (a)

α is a differentiable parametrized curve, regular except at $t = \dfrac{\pi}{2}$.

Solution: Let $x(t) = \sin t, y(t) = \cos t + \log \tan \frac{t}{2}$. We will assume that log is the natural logarithm.

$$x'(t) = \cos t$$
$$y'(t) = -\sin t + \frac{1}{2\tan\frac{t}{2}\cos^2\frac{t}{2}} = -\sin t + \frac{1}{2\sin\frac{t}{2}\cos\frac{t}{2}} = -\sin t + \frac{1}{\sin t}$$

Both functions are well-behaved for $t \in (0, \pi)$.

$$|\alpha'(t)| = \sqrt{\cos^2 t + \sin^2 t + \frac{1}{\sin^2 t} - 2} = \sqrt{\frac{1}{\sin^2 t} - 1} = \sqrt{\frac{1-\sin^2 t}{\sin^2 t}} = \sqrt{\cot^2 t} = |\cot t|$$

And clearly $|\alpha'(t)| \neq 0$, except when $t = \dfrac{\pi}{2}$

Part (b)

The length of the segment of the tangent of the tractrix between the point of tangency and the y axis is constantly equal to 1.

Solution: From the question, we know that the slope of the tangent line is $\pm\cot t$ (parity is irrelevant). Construct a line tangent to $x(t)$ and compute the intersection at the y-axis. Obtain the y-displacement. Let ℓ be the length of the segment.

$$\begin{aligned}\ell &= \sqrt{x^2(t) + (x(t)\cot t)^2} \\ &= \sqrt{\sin^2 t + \cos^2 t} \\ &= 1\end{aligned}$$

Question 5

Let $\alpha : (-1, \infty) \to \mathbb{R}^2$ be given by

$$\alpha(t) = \Big(\frac{3at}{1+t^3}, \frac{3at^2}{1+t^3}\Big)$$

Prove that

Part (a)

For $t = 0$, α is tangent to the x axis.

Solution: $\alpha'(t) = \Big(\dfrac{(3a)(1+t^3) - (3at)(3t^2)}{(1+t^3)^2}, \dfrac{(6at)(1+t^3) - (3at^2)(3t^2)}{(1+t^3)^2}\Big), \alpha'(0) = (3a, 0)$

Part (b)

As $t \to \infty$, $\alpha(t) \to (0, 0)$ and $\alpha'(t) \to (0, 0)$.

Solution: Trivial from limits of rational functions.

Part (c)

Take the curve with the opposite orientation. Now, as $t \to -1$, the curve and its tangent approaches the line $x + y + a = 0$.

Solution: Since $t \in (-1, \infty)$, we have $t \to -1^+$.

$$\lim_{t\to -1^+} \frac{(1+t)^3}{1+t^3} = \lim_{t\to 0^+} \frac{(1+(t-1))^3}{1+(t-1)^3} = \lim_{t\to 0^+} \frac{t^3}{t^3-3t^3+3t} = \lim_{t\to 0^+} \frac{t^2}{t^2-3t+3} = 0$$

Then we can compute the following limit:

$$\begin{aligned}
\lim_{t\to -1} x(t)+y(t) &= \lim_{t\to -1} \frac{3at^2+3at}{1+t^3} \\
&= \lim_{t\to -1} \frac{at^3+3at^2+3at+a}{1+t^3} - a \\
&= \lim_{t\to -1} a\frac{(1+t)^3}{1+t^3} - a \\
&= -a
\end{aligned}$$

Question 6

Let $\alpha(t) = (ae^{bt}\cos t, ae^{bt}\sin t), t \in R$, a and b constants, $a > 0, b < 0$, be a paremetrized curve.

Part (a)

Show that as $t \to \infty$, $\alpha(t)$ approaches the origin 0, spiraling around it. (because of this, the trace of α is called the logarithmic spiral; see Fig. 1-11).

Solution: $\lim_{t\to\infty} x(t) = \lim_{t\to\infty} y(t) = 0$

Part (b)

Show that $\alpha'(t) \to (0,0)$ as $t \to \infty$ and that

$$\lim_{t\to\infty} \int_{t_0}^{t} |\alpha'(t)| dt$$

is finite; that is, α has finite arc length in $[t_0, \infty)$.

Solution: $x'(t) = abe^{bt}\cos t - ae^{bt}\sin t, y'(t) = abe^{bt}\sin t + ae^{bt}\cos t$.

$$\begin{aligned}
\lim_{t\to\infty} \int_{t_0}^{t} |\alpha'(t)| dt &= \lim_{t\to\infty} \int_{t_0}^{t} \sqrt{x'(t)^2+y'(t)^2} dt \\
&= \lim_{t\to\infty} \int_{t_0}^{t} \sqrt{a^2b^2e^{2bt}+a^2e^{2bt}} dt \\
&= a\sqrt{b^2+1} \lim_{t\to\infty} \int_{t_0}^{t} e^{bt} dt \\
&= -\frac{a\sqrt{b^2+1}}{b} e^{bt_0} < \infty
\end{aligned}$$

Question 7

A map $\alpha : I \to \mathbb{R}^3$ is called a curve of class C^k if each of the coordinate functions in the expression $\alpha(t) = (x(t), y(t), z(t))$ has continuous derivatives up to order k. If α is merely continuous, we say that α is of class C^0. A curve α is called simple if the map α is one-to-one. Thus, the curve in Example 3 of Sec. 1-2 is not simple.

Let $\alpha : I \to \mathbb{R}^3$ be a simple curve of class C^0. We say that α has a weak tangent at $t = t_0 \in I$ if the line determined by $\alpha(t_0 + h)$ and $\alpha(t_0)$ has a limit position when $h \to 0$. We say that α has a strong tangent at $t = t_0$ if the line determined by $\alpha(t_0 + h)$ and $\alpha(t_0 + k)$ has a limit position when $h, k \to 0$. Show that

Part (a)

$\alpha(t) = (t^3, t^2)$, $t \in \mathbb{R}$, has a weak tangent but not a strong tangent at $t = 0$.

Solution:

$$\frac{x}{y} = \lim_{h\to 0} \frac{x(h) - x(0)}{y(h) - y(0)} = \lim_{h\to 0} h = 0$$

So the weak tangent is $x = 0$. For the strong tangent, consider taking $h = -k$.

$$\frac{y}{x} = \lim_{h\to 0} \frac{y(h) - y(k)}{x(h) - x(k)} = \lim_{h\to 0} \frac{0}{x(h) - x(k)} = 0$$

Which offers that the limit position is $y = 0$. If the strong tangent exists, then it must be equal to the weak tangent. However, we found a conflicting limit position. Therefore, there is no strong tangent.

Part (b)

If $\alpha : I \to \mathbb{R}^3$ is of class C^1 and regular at $t = t_0$, then it has a strong tangent at $t = t_0$.

Solution: $\alpha(t)$ is regular at $t = t_0$, so $|\alpha'(t_0)| \neq 0$. Then $x'(t_0), y'(t_0), z'(t_0)$ are not all 0. We can obtain the following relations to describe line defined by points at $t = h$ and $t = k$.

$$\frac{x - x(k)}{\frac{x(h)-x(k)}{h-k}} = \frac{y - y(k)}{\frac{y(h)-y(k)}{h-k}} = \frac{z - z(k)}{\frac{z(h)-z(k)}{h-k}}$$

Note that x, y, z are variables defining the tangent line. Consider the limit of the LHS:

$$\lim_{h,k\to t_0} \frac{x - x(k)}{\frac{x(h)-x(k)}{h-k}} = \frac{x - x(t_0)}{x'(t_0)}$$

which clearly exists. By the equivalence across all x, y, z, we must have a strong tangent.

Part (c)

The curve given by

$$\alpha(t) = \begin{cases} (t^2, t^2), & t \geq 0, \\ (t^2, -t^2), & t \leq 0, \end{cases}$$

is of class C^1 but not of class C^2. Draw a sketch of the curve and its tangent vectors.

Solution: $\alpha'(0)$ is a C^0 function.

$$\alpha'(t) = \begin{cases} (2t, 2t), & t \geq 0 \\ (2t, -2t), & t \leq 0 \end{cases}$$

Question 8

Let $\alpha : I \to \mathbb{R}^3$ be a differentiable curve and let $[a,b] \subset I$ be a closed interval. For every partition

$$a = t_0 < t_1 < \cdots < t_n = b$$

of $[a,b]$, consider the sum $\sum_{i=1}^{n} |\alpha(t_i) - \alpha(t_{i-1})| = l(\alpha, P)$, where P stands for the given partition. The norm $|P|$ of a partition P is defined as

$$|P| = \max(t_i - t_{i-1}), \quad i = 1, \ldots, n.$$

Geometrically, $l(\alpha, P)$ is the length of a polygon inscribed in $\alpha([a,b])$ with vertices in $\alpha(t_i)$. The point of the exercise is to show that the arc length of $\alpha([a,b])$ is, in some sense, a limit of lengths of inscribed polygons.

Prove that given $\varepsilon > 0$ there exists $\delta > 0$ such that if $|P| < \delta$ then

$$\left| \int_a^b |\alpha'(t)| dt - l(\alpha, P) \right| < \varepsilon$$

Solution: Let $\varepsilon > 0$ and continue the inequality after applying the triangle inequality.

$$\left| \int_a^b |\alpha'(t)| dt - l(\alpha, P) \right| \leq \left| \int_a^b |\alpha'(t)| dt - \sum_{i=1}^{n} \frac{|\alpha'(t_i)|}{n} \right| + \left| \sum_{i=1}^{n} \frac{|\alpha'(t_i)|}{n} - l(\alpha, P) \right| \tag{1}$$

Observe the RHS. $|\alpha'(t)|$ is a continuous function thus Riemann integrable. Hence, we can always take a Riemann integral approximation such that the absolute difference is very small. More formally, $\forall \varepsilon > 0, \exists N_1$ such that

$$n > N_1 \implies \left| \int_a^b |\alpha'(t)| dt - \sum_{i=1}^{n} \frac{|\alpha'(t_i)|}{n} \right| < \frac{\varepsilon}{2}$$

where we can force $t_i = a + \frac{b-a}{n} i$ without loss of generality. For the second term,

$$\left| \sum_{i=1}^{n} \frac{|\alpha'(t_i)|}{n} - l(\alpha, P) \right| = \left| \sum_{i=1}^{n} \frac{1}{n} |\alpha'(t_i)| - \sum_{i=1}^{n} \frac{1}{n} \frac{|\alpha(t_i)| - |\alpha(t_{i-1})|}{\frac{1}{n}} \right|$$
$$= \sum_{i=1}^{n} \frac{1}{n} \left| |\alpha'(t_i)| - \frac{|\alpha(t_i)| - |\alpha(t_{i-1})|}{\frac{1}{n}} \right|$$

By the mean value theorem, $\forall i, \exists s_i \in (t_{i-1}, t_i)$ such that $\frac{|\alpha(t_i)| - |\alpha(t_{i-1})|}{\frac{1}{n}} = |\alpha'(s_i)|$. Furthermore, we have $\forall \varepsilon > 0, \exists N_2$ such that if $n > N_2$, then $\Big| |\alpha'(t_i)| - |\alpha'(s_i)| \Big| < \frac{\varepsilon}{2}$.

$$= \sum_{i=1}^{n} \frac{1}{n} \big| |\alpha'(t_i)| - |\alpha'(s_i)| \big| \leq \sum_{i=1}^{n} \frac{1}{n} \frac{\varepsilon}{2} \leq \frac{\varepsilon}{2}$$

Revisiting (1), take $\delta < \frac{1}{N}$ where $N > \max\{N_1, N_2\}$, and we have bounded the RHS by ε. This concludes the proof. ■

Question 9

Part (a)

Let $\alpha : I \to \mathbb{R}^3$ be a curve of class C^0 (cf. Exercise 7). Use the approximation by polygons described in Exercise 8 to give a reasonable definition of arc length of α.

Solution: Trivial by $l(\alpha, P)$ defined in question 8 and selection of partition P.

Part (b)

A Nonrectifiable Curve. The following example shows that, with any reasonable definition, the arc length of a C^0 curve in a closed interval may be unbounded. Let $\alpha : [0,1] \to \mathbb{R}^2$ be given as $\alpha(t) = (t, t\sin(\pi/t))$ if $t \neq 0$, and $\alpha(0) = (0,0)$. Show, geometrically, that the arc length of the portion of the curve corresponding to $1/(n+1) \leq t \leq 1/n$ is at least $2/(n+\frac{1}{2})$. Use this to show that the length of the curve in the interval $1/N \leq t \leq 1$ is greater than $2\sum_{n=1}^{N} 1/(n+1)$, and thus it tends to infinity as $N \to \infty$.

Solution: $\alpha'(t) = \left(1, \sin\frac{\pi}{t} - \frac{\pi}{t}\cos\frac{\pi}{t}\right)$.

$$|\alpha'(t)| = \sqrt{1 + \left(\sin\frac{\pi}{t} - \frac{\pi}{t}\cos\frac{\pi}{t}\right)^2} \geq \left|\sin\frac{\pi}{t} - \frac{\pi}{t}\cos\frac{\pi}{t}\right|$$

We will take an alternative approach with modified bounds $t \in \left(\frac{1}{n+\frac{1}{2}}, \frac{1}{n}\right)$.

$$\begin{aligned}
\int_{\frac{1}{n+1}}^{\frac{1}{n}} |\alpha'(t)| dt &\geq \int_{\frac{1}{n+\frac{1}{2}}}^{\frac{1}{n}} \left|\sin\frac{\pi}{t} - \frac{\pi}{t}\cos\frac{\pi}{t}\right| dt \\
&\geq \left|\int_{\frac{1}{n+\frac{1}{2}}}^{\frac{1}{n}} \left(\sin\frac{\pi}{t} - \frac{\pi}{t}\cos\frac{\pi}{t}\right) dt\right| \\
&= \left|\left(t\sin\frac{\pi}{t}\right)_{\frac{1}{n+\frac{1}{2}}}^{\frac{1}{n}}\right| \\
&= \left|\frac{1}{n+\frac{1}{2}}\right|
\end{aligned}$$

And thus,

$$\int_{\frac{1}{N}}^{1} |\alpha'(t)| dt = \sum_{n=1}^{N-1} \int_{\frac{1}{n+1}}^{\frac{1}{n}} |\alpha'(t)| dt \geq \sum_{n=1}^{N-1} \left|\frac{1}{n+\frac{1}{2}}\right| \geq \sum_{n=1}^{N-1} \left|\frac{1}{n+1}\right|.$$

Question 10

(Straight Lines as Shortest.) Let $\alpha : I \to \mathbb{R}^3$ be a parametrized curve. Let $[a,b] \subset I$ and set $\alpha(a) = p$, $\alpha(b) = q$.

Part (a)

Show that, for any constant vector v, $|v| = 1$,

$$(q-p) \cdot v = \int_a^b \alpha'(t) \cdot v \, dt \leq \int_a^b |\alpha'(t)| \, dt.$$

Solution: Apply the fundamental theorem of calculus and Cauchy-Schwarz.

$$(q-p) \cdot v = (\alpha(b) - \alpha(a)) \cdot v = \int_a^b \alpha'(t) dt \cdot v = \int_a^b (\alpha'(t) \cdot v) dt \leq \int_a^b |\alpha'(t)||v| dt \leq \int_a^b |\alpha'(t)| dt$$

Part (b)

Set $v = \dfrac{q-p}{|q-p|}$ and show that $|\alpha(b) - \alpha(a)| \leq \int_a^b |\alpha'(t)| dt$; that is, the curve of shortest length from $\alpha(a)$ to $\alpha(b)$ is the straight line joining these two points.

Solution: Taking v as mentioned, $(q-p) \cdot v = \dfrac{|q-p|^2}{|q-p|} = |q-p| = |\alpha(b) - \alpha(a)|$. And the result follows from (a).